Name

Student Number

Gosford High School

Trial HSC 2023

Mathematics Extension 2

General Instructions

- Reading time – 10 minutes
- Working time – 3 hours
- Write using black or blue pen
- Calculators approved by NESA may be used
- A reference sheet is provided
- In Section II, show relevant mathematical reasoning and/ or calculations

Total Marks

100

Section I – 10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section
- Answer questions on the Multiple-choice answer sheet

Section II – 90 marks

- Attempt Questions 11–16
- Allow about 2 hours 45 minutes for this section
- Answer questions on the answer sheet provided for each question

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Section I
10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Shade the best response on the multiple-choice answer sheet.

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1. Find $\int_{-\pi/2}^{\pi/2} \sin^5 x \ dx$

(A) 0

(B) 2

(C) $\frac{\pi}{8}$

(D) $\frac{3\pi}{8}$

2. Express $-2\sqrt{3} + 2\mathrm{i}$ in modulus/argument form.

(A) $4\left(\cos\left(\frac{7\pi}{6}\right) + \sin\left(\frac{7\pi}{6}\right)\right)$

(B) $3\left(\cos\left(\frac{7\pi}{6}\right) + \sin\left(\frac{7\pi}{6}\right)\right)$

(C) $4\left(\cos\left(\frac{5\pi}{6}\right) + \sin\left(\frac{5\pi}{6}\right)\right)$

(D) $3\left(\cos\left(\frac{5\pi}{6}\right) + \sin\left(\frac{5\pi}{6}\right)\right)$

3. Which of the following points lies on the line described by the vector equation:

$$\underset{\sim}{r} = \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 3 \\ 1 \end{pmatrix}$$

(A) $\begin{pmatrix} -3 \\ 9 \\ 1 \end{pmatrix}$

(B) $\begin{pmatrix} -3 \\ -8 \\ -3 \end{pmatrix}$

(C) $\begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$

(D) $\begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix}$

4. If the vectors $\underset{\sim}{u} = \lambda \underset{\sim}{i} + \lambda \underset{\sim}{j} - 2\underset{\sim}{k}$ and $\underset{\sim}{v} = \lambda \underset{\sim}{i} - 2\underset{\sim}{j} + 4\underset{\sim}{k}$ are perpendicular, then

(A) $\lambda = -2$ or $\lambda = 4$

(B) $\lambda = -4$ or $\lambda = 2$

(C) $\lambda = -4$ or $\lambda = -2$

(D) $\lambda = 2$ or $\lambda = 4$

5. Consider the statement:

For any function $f(x)$, $f(x)$ has a stationary point of inflection at $x = c \Rightarrow f''(c) = 0$.

Which of the following is correct?

(A) The contrapositive statement is false and the converse statement is false.

(B) The contrapositive statement is false and the converse statement is true.

(C) The contrapositive statement is true and the converse statement is false.

(D) The contrapositive statement is true and the converse statement is true.

6. P, Q and R are collinear points with position vectors $\underset{\sim}{p}, \underset{\sim}{q}$ and $\underset{\sim}{r}$. P is between Q and R.
If $|QR| = 3|PR|$, then $\underset{\sim}{r}$ has the position vector:

(A) $\underset{\sim}{r} = \frac{1}{2}\underset{\sim}{p} - \frac{3}{2}\underset{\sim}{q}$

(B) $\underset{\sim}{r} = \frac{3}{2}\underset{\sim}{p} - \frac{1}{2}\underset{\sim}{q}$

(C) $\underset{\sim}{r} = \frac{3}{2}\underset{\sim}{p} + \frac{1}{2}\underset{\sim}{q}$

(D) $\underset{\sim}{r} = -\frac{1}{2}\underset{\sim}{p} + \frac{3}{2}\underset{\sim}{q}$

7. Given that $|z + 3| = 2$ and $\arg(z + 3) = \frac{5\pi}{6}$, which of the following is an expression for $z + 3$?

(A) $\frac{\sqrt{3}}{2} + \frac{1}{2}i$

(B) $-\frac{\sqrt{3}}{2} + \frac{1}{2}i$

(C) $\sqrt{3} + i$

(D) $-\sqrt{3} + i$

8. Which of the following is an expression for $\int \frac{3\,dx}{x^2+2x+2}$?

(A) $\frac{1}{3}\tan^{-1}(x+1)+c$

(B) $3\tan^{-1}(x+1)+c$

(C) $\frac{1}{3}\tan^{-1}(x-1)+c$

(D) $3\tan^{-1}(x-1)+c$

9. What is the remainder when $P(z) = 13z^4 - 7z + 3$ is divided by $z - i$?

(A) $10 + 7i$

(B) $16 + 7i$

(C) $10 - 7i$

(D) $16 - 7i$

10. Given that x and y are real numbers, which of the following is a TRUE statement?

(A) $\forall y\, \exists x: x^2 - y^2 = x$

(B) $\forall y\, \exists x: x^2 - y^2 = y$

(C) $\forall y\, \exists x: x^2 + y^2 = x$

(D) $\forall y\, \exists x: x^2 + y^2 = y$

Section II
90 marks
Attempt Questions 11-16
Allow about 2 hours 45 minutes for this section

Answer in the appropriate booklet provided

Question 11. (15 marks) **MARKS**

Answer in the booklet labelled *Question 11*

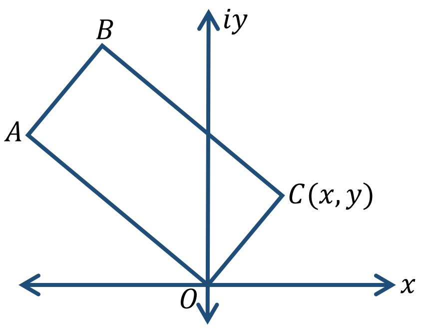
(a) If $A = 3 - 4i$ and $B = 1 + i$, evaluate the following:

(i) $A - B$ **1**

(ii) $\frac{A}{B}$ **2**

(iii) $\sqrt{A}$ **3**

b) On the Argand diagram shown $OABC$ is a rectangle with the length OA being twice OC.



Find the complex number represented by

(i) OA **1**

(ii) OB **1**

(iii) BC **1**

(c) Given that $C = 1 + \sqrt{3}\,i$,

(i) Write C in modulus-argument form. **2**

(ii) Hence find C^6 **2**

(d) On an Argand diagram sketch the region where

$|z - 1| \leq \sqrt{2}$ and $0 \leq arg\,(z + i) \leq \frac{\pi}{4}$ both hold. **2**

Question 12 (15 Marks) **MARKS**

Answer in the booklet labelled *Question 12*

(a) Find $\int x\cos(x^2)\, e^{\sin(x^2)} dx$ **2**

(b) Find $\int_0^1 \sin^{-1} x \ dx$ by using integration by parts. **3**

(c) Find $\int \frac{2x^2+4x-3}{x+1} dx$ **3**

(d) (i) Find the real numbers A and B such that: **2**

$$\frac{3x+2}{x^2-4} \equiv \frac{A}{x+2} + \frac{B}{x-2}$$

(ii) Hence show that $\int_3^5 \frac{3x+2}{x^2-4} dx \quad = \ln\left(\frac{63}{5}\right)$ **2**

(e) Find $\int (1+2x^2)e^{x^2} \ dx$ **3**

Question 13 (15 Marks) **Marks**

Answer in the booklet labelled *Question 13*

(a) (i) Given that $z = e^{i\theta}$, show that $z^k + z^{-k} = 2\cos(k\theta)$ **1**

(ii) By expanding $(z + z^{-1})^4$ or otherwise, show that $\cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$ **3**

(b) Prove by mathematical induction that for all integers $n > 1$, $12^n > 7^n + 5^n$. **3**

(c) Find the Cartesian equation of the sphere with centre $\underset{\sim}{c} = -\underset{\sim}{i} + 2\underset{\sim}{j} - \underset{\sim}{k}$ which passes through the point $\underset{\sim}{a} = 2\underset{\sim}{i} + \underset{\sim}{j} + \underset{\sim}{k}$ **3**

(d) Let $a, b, n \in \mathbb{N}$.

(i) Prove that if $n = ab$, then $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$. **3**

(ii) Hence, show that 97 is a prime number. **2**

Question 14 (15 Marks) **MARKS**

Answer in the booklet labelled *Question 14*

(a) Goldbach's conjecture is that every even integer greater than two can be expressed as the sum of two primes. To date, no one has been able to prove this, although it has been verified for all integers less than 4×10^{18}.

(i) Prove that 101 cannot be written as the sum of two prime numbers. **2**

(ii) Assuming that Goldbach's conjecture is true, prove that every odd integer greater than 5 can be written as the sum of three prime numbers. **3**

(b) Evaluate $\int_0^{\pi/2} \frac{d\theta}{1+\sin\theta+\cos\theta}$ **4**

(c) (i) Show that if $I_n = \int \tan^n x\, dx$, then $I_n = \frac{1}{n-1}\tan^{n-1} x - I_{n-2}$ **3**

(ii) Hence evaluate $\int_0^{\pi/4} \tan^3 x\, dx$. **3**

Question 15 (15 Marks)

Answer in the booklet labelled *Question 15*

(a) Given the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$ satisfy $\underset{\sim}{u} + \underset{\sim}{v} = 17\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ and $\underset{\sim}{u} - \underset{\sim}{v} = \underset{\sim}{i} + 9\underset{\sim}{j} - 4\underset{\sim}{k}$, find the acute angle between the vectors $\underset{\sim}{u}$ and $\underset{\sim}{v}$. **3**

(b) D is the midpoint of the side BC of a triangle ABC. Using vectors, show that:

$$\left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AC}\right|^2 = 2\left(\left|\overrightarrow{AD}\right|^2 + \left|\overrightarrow{BD}\right|^2\right)$$ **4**

(c) With respect to a fixed origin O, the lines L_1 and L_2 have equations $\underset{\sim}{r}_1 = \begin{pmatrix} 11 \\ 2 \\ 17 \end{pmatrix} + \lambda \begin{pmatrix} -2 \\ 1 \\ -4 \end{pmatrix}$ and $\underset{\sim}{r}_2 = \begin{pmatrix} -5 \\ 11 \\ p \end{pmatrix} + \mu \begin{pmatrix} q \\ 2 \\ 2 \end{pmatrix}$ where λ and μ are scalar parameters and p and q are constants.

i) If L_1 and L_2 intersect at right angles, show that $q = -3$ **1**

ii) Find the value of p. **3**

iii) Find the coordinates of the point of intersection. **2**

(d) Find $\int \frac{3dx}{x^2 - 6x + 13}$ **2**

Question 16 (15 Marks) **MARKS**

Answer in the booklet labelled Question 16

(a) Find the fourth roots of $2 + 2\sqrt{3}i$ **4**

(b) Relative to the origin O, the points A, B, C and D have position vectors given respectively by $-4\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k}, 4\underset{\sim}{i} + c\underset{\sim}{j} + 6\underset{\sim}{k}, 4\underset{\sim}{i} - \underset{\sim}{j} - \underset{\sim}{k}$ and $2\underset{\sim}{j} - 6\underset{\sim}{k}$

i) Given that the line AC is perpendicular to the line BD, determine the value of c. **2**

ii) Hence find the position vector of E, the point of intersection of the lines AC and BD. **3**

(c) i) By considering the graph of $y = \frac{1}{x\sqrt{x}}$, or otherwise, show that for all positive integers $k \geq 1$, $\frac{1}{(k+1)\sqrt{k+1}} < \frac{2}{\sqrt{k}} - \frac{2}{\sqrt{k+1}}$ **2**

ii) Hence use mathematical induction to show that for all positive integers $n \geq 2$,
$\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \cdots + \frac{1}{n\sqrt{n}} < 3 - \frac{2}{\sqrt{n}}$ **4**

End of Exam

Year 12 Extension 2 Trial 2023 – Solutions

Section I

1	2	3	4	5	6	7	8	9	10
A	**C**	**C**	**A**	**C**	**B**	**D**	**B**	**D**	**A**

Question 1 – **A** is correct.

Since $y = \sin^5 x$ is an odd function then $\displaystyle\int_{-\pi/2}^{\pi/2} \sin^5 x \; dx \quad = 0$

Question 2 – **C** is correct.

Let $z = -2\sqrt{3} + 2i$. Then $|z| = \sqrt{\left(-2\sqrt{3}\right)^2 + 2^2}$

$= 4$

Also $\arg(z) \quad = \tan^{-1}\left(-\frac{2}{2\sqrt{3}}\right)$

$= \frac{5\pi}{6}$ as z is in the second quadrant.

Question 3 – **C** is correct.

If $\lambda = -1,\ \underset{\sim}{r} = \begin{pmatrix} 1 - 1 \times -2 \\ 2 - 1 \times 3 \\ -1 - 1 \times 1 \end{pmatrix} = \begin{pmatrix} 3 \\ -1 \\ -2 \end{pmatrix}$

Question 4 – **A** is correct.

$\underset{\sim}{u} \cdot \underset{\sim}{v} \quad = \lambda \times \lambda + \lambda \times -2 + -2 \times 4$

$= \lambda^2 - 2\lambda - 8$

$= 0 \ \Rightarrow \ \lambda = -2$ or $\lambda = 4$

Question 5 – **C** is correct.

Contrapositive: $f''(c) \neq 0 \ \Rightarrow f(x)$ *does not have a stationary point of inflection at* $x = c$. TRUE

Converse: $f''(c) = 0 \ \Rightarrow f(x)$ *has a stationary point of inflection at* $x = c$. FALSE

Question 6 – **B** is correct.

Now $|QR| = 3|PR| = -\underset{\sim}{q} + \underset{\sim}{r}$ and $|PR| = -\underset{\sim}{p} + \underset{\sim}{r}$

Hence $-\underset{\sim}{q} + \underset{\sim}{r} = -3\underset{\sim}{p} + 3\underset{\sim}{r}$

$\therefore 2\underset{\sim}{r} = 3\underset{\sim}{p} - \underset{\sim}{q} \Rightarrow \underset{\sim}{r} = \frac{3}{2}\underset{\sim}{p} - \frac{1}{2}\underset{\sim}{q}$

Question 7 – **D** is correct.

$$\begin{aligned} z + 3 &= 2\left(\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right) \\ &= -\frac{2\sqrt{3}}{2} + \frac{2}{2}i \\ &= -\sqrt{3} + i \end{aligned}$$

Question 8 – **B** is correct.

$$\begin{aligned} \int \frac{3\,dx}{x^2+2x+2} &= \int \frac{3\,dx}{(x+1)^2+1} \\ &= 3\tan^{-1}(x+1) + c \end{aligned}$$

Question 9 – **D** is correct.

$$\begin{aligned} P(i) &= 13(i)^4 - 7i + 3 \\ &= 13 - 7i + 3 \\ &= 16 - 7i \end{aligned}$$

The remainder is $16 - 7i$

Question 10 – **A** is correct.

$\forall y,\ x^2 - y^2 = x$ is an equation with real solutions given by $y = \pm\sqrt{x^2 - x}$
B is incorrect. $y = -\frac{1}{2}$ provides a counter example since there is no real x such that $x^2 = \frac{1}{4} - \frac{1}{2}$
C is incorrect. $y = 1$ provides a counter example since $x^2 + 1 = x$ which has no real solutions, as $\triangle = -3$.
D is incorrect. $y = 2$ provides a counter example since there is no real x such that $x^2 + 4 = 2$

Section II

Question 11.

(a) (i) $A - B = (3 - 4i\ ) - (1 + i) = 2 - 5i$

Marking criteria	Marks
Correct answer	1

(a) (ii) $\frac{A}{B} = \frac{3-4i}{1+i} \times \frac{1-i}{1-i} = \frac{3-7i-4}{1+1} = \frac{-1-7i}{2}$

Marking criteria	Marks
Correct answer	2
Multiplying by the conjugate or equivalent merit	1

(a) (iii) Let $(x + iy)^2 = 3 - 4i$. Then $x^2 - y^2 + 2ixy = 3 - 4i$.

Hence $x^2 - y^2 = 3$ and $2xy = -4 \Rightarrow xy = -2 \Rightarrow y = -\frac{2}{x}$

Hence $x^2 - \frac{4}{x^2} - 3 = 0 \Rightarrow x^4 - 3x^2 - 4 = 0 \Rightarrow x^2 = 4\ or - 1$.

Since $x \in \mathbb{R}, x = 2, y = -1$ or $x = -2, y = 1$

Hence $\sqrt{A} = \pm 2 \mp i$

Marking criteria	Marks
Correct answer	3
Solving simultaneous equations in one variable or equivalent merit.	2
Setting up simultaneous equations or equivalent merit.	1

(b) (i) $OA = 2(-y + ix)$

Marking criteria	Marks
Correct answer	1

(b) (ii) $OB = OA + AB = -2y + 2ix + x + iy = (x - 2y) + i(2x + y)$

Marking criteria	Marks
Correct answer	1

(b) (iii) $BC = -OA = 2y - 2xi$

Marking criteria	Marks
Correct answer	1

(c) (i)

$$C = 1 + \sqrt{3}\, i$$
$$= 2\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\, i\right)$$
$$= 2\left(\cos\left(\frac{\pi}{3}\right) + i \sin\left(\frac{\pi}{3}\right)\right)$$

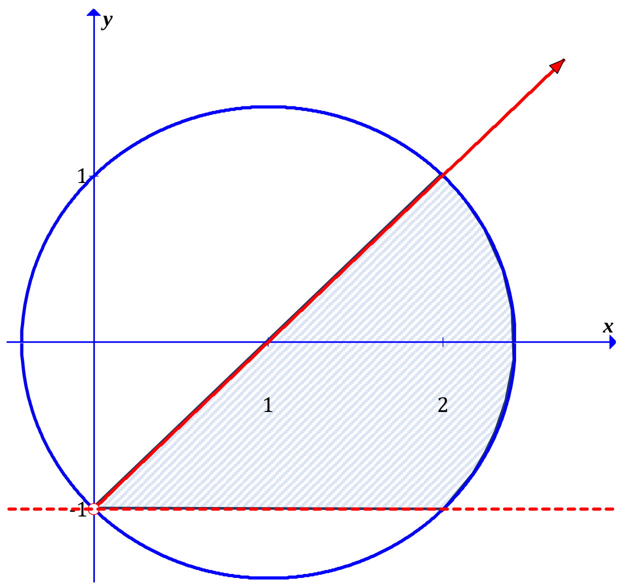
Marking criteria	Marks
Correct answer	2
Finding modulus or argument or equivalent merit.	1

(c) (ii)

$$C^6 = 2^6\left(\cos\left(\frac{6\pi}{3}\right) + i \sin\left(\frac{6\pi}{3}\right)\right)$$
$$= 64(\cos(2\pi) + i \sin(2\pi))$$
$$= 64$$

Marking criteria	Marks
Correct answer	2
Using De Moivre's Theorem or equivalent merit.	1

(d)



Marking criteria	Marks
Correct graphs and shading	2
One correct graph for circle or argument or equivalent merit.	1

Question 12

(a) Now $\frac{d}{dx}\left(e^{\sin(x^2)}\right) = 2x\cos(x^2)e^{\sin(x^2)}$

$$\therefore \int x\cos(x^2)\, e^{\sin(x^2)} dx = \frac{1}{2}e^{\sin(x^2)} + C$$

Marking criteria	Marks
Correct answer	2
Using $\int f'(x)e^{f(x)}dx = e^{f(x)} + C$ or equivalent merit.	1

(b)
$$\int_0^1 \sin^{-1} x \; dx = \int_0^1 1 \times \sin^{-1} x \; dx$$
$$= \Big[x\sin^{-1} x \Big]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\, dx$$
$$= \Big[x\sin^{-1} x + \sqrt{1-x^2} \Big]_0^1$$
$$= \frac{\pi}{2} - 1$$

Marking criteria	Marks
Correct answer	3
Obtaining $\Big[x\sin^{-1} x + \sqrt{1-x^2} \Big]_0^1$ or equivalent merit.	2
Obtaining $\Big[x\sin^{-1} x \Big]_0^1 - \int_0^1 \frac{x}{\sqrt{1-x^2}}\, dx$ or equivalent merit.	1

(c)
$$\int \frac{2x^2+4x-3}{x+1} dx = \int \frac{(2x+2)(x+1)-5}{x+1} dx$$
$$= \int \left(2x + 2 - \frac{5}{x+1}\right) dx$$
$$= x^2 + 2x - 5\ln(x+1) + C$$

Marking criteria	Marks
Correct answer	3
Obtaining $2x + 2 - \frac{5}{x+1}$ or equivalent merit.	2
Attempt to divide $2x^2 + 4x - 3$ by $x + 1$ or equivalent merit.	1

(d) (i) Let $\frac{3x+2}{x^2-4} \equiv \frac{A}{x+2} + \frac{B}{x-2}$. Then $A(x-2) + B(x+2) = 3x + 2$

Now if $x = 2, 4B = 8 \quad \Rightarrow B = 2$

Also if $x = -2, -4A = -4 \quad \Rightarrow A = 1$

Marking criteria	Marks
Finding correct values of A and B	2
Obtaining $A(x-2) + B(x+2) = 3x + 2$ or equivalent merit.	1

(d) (ii) $\int_3^5 \frac{3x+2}{x^2-4} dx$

$$= \int_3^5 \left(\frac{1}{x+2} + \frac{2}{x-2}\right) dx$$

$$= \Big[\ln(x+2) \quad + \quad 2\ln(x-2)\Big]_3^5$$

$$= \ln 7 + 2\ln 3 - \ln 5 - 2\ln 1$$

$$= \ln\left(\frac{7\times 3^2}{5}\right) = \ln\left(\frac{63}{5}\right)$$

Marking criteria	Marks
Correct answer	2
Obtaining $A\Big[\ln(x+2) \quad + \quad 2\ln(x-2)\Big]_3^5$ or equivalent merit.	1

(e) Consider $\int 2x^2 e^{x^2}\, dx$

$$= \int x \times 2xe^{x^2}\, dx$$

$$= xe^{x^2} + c - \int e^{x^2}\, dx$$

$$\therefore \int (1 + 2x^2)e^{x^2}\, dx \quad = xe^{x^2} + c$$

Marking criteria	Marks
Correct solution	3
Correctly performs the integration by parts or equivalent merit.	2
Correct separation for integration by parts or equivalent merit.	1

Question 13

(a) (i) $z^k + z^{-k} = (\cos(k\theta) + i\sin(k\theta)) + (\cos(-k\theta) + i\sin(-k\theta))$

$= (\cos(k\theta) + i\sin(k\theta)) + (\cos(k\theta) - i\sin(k\theta))$

$= 2\cos(k\theta)$

Marking criteria	Marks
Correct solution	1

(a) ii) Now $(z + z^{-1})^4 = z^4 + 4z^3z^{-1} + 6z^2z^{-2} + 4zz^{-3} + z^{-4}$

Hence $(2\cos\theta)^4 = (z^4 + z^{-4}) + 4(z^2 + z^{-2}) + 6$

Hence $16\ \cos^4\theta = 2\cos 4\theta + 8\cos 2\theta + 6$

$\therefore \cos^4\theta = \frac{1}{8}\cos 4\theta + \frac{1}{2}\cos 2\theta + \frac{3}{8}$

Marking criteria	Marks
Correct answer	3
Use of part (i) to simplify the expanded quartic or equivalent merit	2
Expands the quartic or equivalent merit	1

(b) $n = 2: LHS = 12^2 = 144,\ RHS = 7^2 + 5^2 = 74$

Hence it is true for $n = 2$.

Suppose that there is a k such that $12^k > 7^k + 5^k$.

Then $12^{k+1} = 12(12^k)$

$> 12(7^k + 5^k) = 12(7^k) + 12(5^k)$

$> 7(7^k) + 5(5^k)$ since $k \in \mathbb{Z}^+$

$= 7^{k+1} + 5^{k+1}$

Hence by the principle of mathematical induction $12^n > 7^n + 5^n$ for $n > 1$.

Marking criteria	Marks
Correct proof	3
Shows some relevant working toward proving the case for $n = k + 1$ or equivalent merit	2
Establishes the result for $n = 2$ or equivalent merit	1

(c) $\overrightarrow{CA} = \overrightarrow{CO} + \overrightarrow{OA}$

$$= \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 \\ -1 \\ 2 \end{pmatrix}$$

$$\left|\overrightarrow{CA}\right| = \sqrt{3^2 + (-1)^2 + 2^2}$$

$$= \sqrt{14}$$

Hence the equation is $(x+1)^2 + (y-2)^2 + (z+1)^2 = 14$

Marking criteria	Marks
Correct answer	3
Finds the radius of the sphere or equivalent merit	2
Finds $\overrightarrow{CA}$ or equivalent merit	1

(d) i) Suppose that if $n = ab$, then $a > \sqrt{n}$ and $b > \sqrt{n}$

Then $ab > \sqrt{n} \times \sqrt{n} = n$

This is a contradiction and so if $n = ab$, hence $a \le \sqrt{n}$ or $b \le \sqrt{n}$.

Marking criteria	Marks
Correct proof	3
Establishing a contradiction or equivalent merit	2
Correct supposition or equivalent merit	1

(d) ii) If $97 = ab$ where $a, b \in \mathbb{Z}$ and neither are 1 or 97, then $a \le \sqrt{97} \approx 9.8$ and since 97 is not divisible by $2, 3, 5$ or 7 (it is not even, the sum of the digits is not divisible by 3, it doesn't end in 5 or 0 and $7 \times 13 = 91$), then 97 is not divisible by any integer less than $\sqrt{97}$.

Hence, 97 is a prime number.

Marking criteria	Marks
Correct proof	2
Establishing that 97 is not divisible by $2, 3, 5$ or 7 or equivalent merit	1

Question 14

(a) (i) Since the sum of two odd numbers is an even number, the only way that 101 can be written as the sum of two prime numbers is if one of them is even.

Since 2 is the only even prime number and $99 = 3 \times 33$ is not prime, then 101 can not be written as the sum of two prime numbers.

Marking criteria	Marks
Correct proof	2
Correctly determining that one of the primes must be 2 or equivalent merit.	1

(a) (ii) If $n = 2k + 1$ is an odd number greater than 5 then $n - 3$ is an even number greater than 2.

Now $n = n - 3 + 3 = 2k + 1 - 3 + 3 = 2(k - 1) + 3$.

By Goldbach's conjecture, $2(k - 1)$ can be expressed as the sum of two primes, and since 3 is prime then n can be expressed as the sum of three primes.

Marking criteria	Marks
Correct proof	3
Making use of Goldbach's conjecture or equivalent merit	2
Correctly determining that n is the sum of an even number plus 3 or equivalent merit.	1

(b) Let $t = \tan\left(\frac{\theta}{2}\right)$, then $\frac{dt}{d\theta} \quad = \frac{1}{2}\sec^2\left(\frac{\theta}{2}\right) \quad = \frac{1}{2}\left(1 + \tan^2\left(\frac{\theta}{2}\right)\right) \quad = \frac{1}{2}(1 + t^2)$

$\therefore \frac{d\theta}{dt} = \frac{2}{1+t^2} \;\Rightarrow\; d\theta = \frac{2dt}{1+t^2}$

Also when $\theta = \frac{\pi}{2}$, $t = 1$ and at $\theta = 0, t = 0$

Hence $\int_0^{\pi/2} \frac{d\theta}{1+\sin\theta+\cos\theta} \quad = \int_0^1 \frac{\frac{2}{1+t^2}dt}{1+\frac{2t}{1+t^2}+\frac{1-t^2}{1+t^2}} \quad = \int_0^1 \frac{2dt}{1+t^2+2t+1-t^2} \quad = \int_0^1 \frac{dt}{1+t}$

$$= \Big[\ln|1+t|\Big]_0^1 \quad = \ln 2$$

Marking criteria	Marks
Correct answer	4
Correct integrand in terms of t or equivalent merit	3
Correct use of the t formulae or equivalent merit	2
Incomplete use of the t formulae or equivalent merit	1

(c) (i) $I_n = \int \tan^n x \, dx$

$$= \int \tan^2 x \times \tan^{n-2} x \, dx$$

$$= \int (\sec^2 x - 1) \tan^{n-2} x \, dx$$

$$= \int \sec^2 x \tan^{n-2} x \, dx \quad - \int \tan^{n-2} x \, dx$$

$$= \frac{1}{n-1} \tan^{n-1} x - I_{n-2}$$

Marking criteria	Marks
Correct proof	3
Correctly expansion or equivalent merit	2
Obtaining $\tan^2 x \times \tan^{n-2} x$ or equivalent merit	1

(c) (ii) $\int_0^{\pi/4} \tan^3 x \, dx$

$$= \left[\frac{1}{2}\tan^2 x\right]_0^{\pi/4} - \int_0^{\pi/4} \tan x \, dx$$

$$= \frac{1}{2} - \left[\ln|\cos x| \right]_0^{\pi/4}$$

$$= \frac{1}{2} + \ln\left(\frac{1}{\sqrt{2}}\right)$$

Marking criteria	Marks
Correct answer	3
Correct integral of $\tan x$ or equivalent merit	2
Correct use of recurrence relation or equivalent merit	1

Question 15

(a) $\underset{\sim}{u} + \underset{\sim}{v} = 17\underset{\sim}{i} - \underset{\sim}{j} + 2\underset{\sim}{k}$ ①

$\underset{\sim}{u} - \underset{\sim}{v} = \underset{\sim}{i} + 9\underset{\sim}{j} - 4\underset{\sim}{k}$ ②

① + ② $2\underset{\sim}{u} = 18\underset{\sim}{i} + 8\underset{\sim}{j} - 2\underset{\sim}{k}$

① − ② $2\underset{\sim}{v} = 16\underset{\sim}{i} - 10\underset{\sim}{j} + 6\underset{\sim}{k}$

$$\therefore \ \underset{\sim}{u} = \begin{pmatrix} 9 \\ 4 \\ -1 \end{pmatrix} \text{ and } \underset{\sim}{v} = \begin{pmatrix} 8 \\ -5 \\ 3 \end{pmatrix}$$

$$\cos\theta = \frac{\underset{\sim}{u}\cdot\underset{\sim}{v}}{|\underset{\sim}{u}||\underset{\sim}{v}|} = \frac{72-20-3}{\sqrt{9^2+4^2+1^2}\times\sqrt{8^2+5^2+3^2}}$$

$$= \frac{1}{2}$$

Hence the acute angle between the vectors is $\frac{\pi}{3}$

Marking criteria	Marks
Correct answer	3
Uses the geometric formula for the dot product or equivalent merit	2
Finds $\underset{\sim}{u}$ and $\underset{\sim}{v}$ or equivalent merit	1

(b) Let O be the origin and A, B, C have position vectors $\underset{\sim}{a}, \underset{\sim}{b}, \underset{\sim}{c}$ respectively.

Then $\overrightarrow{AB} = \underset{\sim}{b} - \underset{\sim}{a}$ and $\overrightarrow{AC} = \underset{\sim}{c} - \underset{\sim}{a}$

Hence $\overrightarrow{BD} = \frac{1}{2}(\underset{\sim}{c} - \underset{\sim}{b}) \Rightarrow \overrightarrow{OD} = \overrightarrow{OB} + \overrightarrow{BD} = \underset{\sim}{b} + \frac{1}{2}(\underset{\sim}{c} - \underset{\sim}{b}) = \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c})$

Hence $\overrightarrow{AD} = \frac{1}{2}(\underset{\sim}{b} + \underset{\sim}{c}) - \underset{\sim}{a}$

$$\begin{aligned}
2\left(\left|\overrightarrow{AD}\right|^2 + \left|\overrightarrow{BD}\right|^2\right) &= 2\left(\left(\tfrac{1}{2}(\underset{\sim}{b}+\underset{\sim}{c}) - \underset{\sim}{a}\right)\cdot\left(\tfrac{1}{2}(\underset{\sim}{b}+\underset{\sim}{c}) - \underset{\sim}{a}\right) + \left(\tfrac{1}{2}(\underset{\sim}{c}-\underset{\sim}{b})\right)\cdot\left(\tfrac{1}{2}(\underset{\sim}{c}-\underset{\sim}{b})\right)\right) \\
&= 2\left(\tfrac{1}{4}(\underset{\sim}{b}+\underset{\sim}{c})\cdot(\underset{\sim}{b}+\underset{\sim}{c}) - \underset{\sim}{a}\cdot(\underset{\sim}{b}+\underset{\sim}{c}) + \underset{\sim}{a}\cdot\underset{\sim}{a} + \tfrac{1}{4}\underset{\sim}{c}\cdot\underset{\sim}{c} - \tfrac{1}{2}\underset{\sim}{b}\cdot\underset{\sim}{c} + \tfrac{1}{4}\underset{\sim}{b}\cdot\underset{\sim}{b}\right) \\
&= 2\left(\tfrac{1}{4}\underset{\sim}{b}\cdot\underset{\sim}{b} + \tfrac{1}{2}\underset{\sim}{b}\cdot\underset{\sim}{c} + \tfrac{1}{4}\underset{\sim}{c}\cdot\underset{\sim}{c} - \underset{\sim}{a}\cdot\underset{\sim}{b} - \underset{\sim}{a}\cdot\underset{\sim}{c} + \underset{\sim}{a}\cdot\underset{\sim}{a} + \tfrac{1}{4}\underset{\sim}{c}\cdot\underset{\sim}{c} - \tfrac{1}{2}\underset{\sim}{b}\cdot\underset{\sim}{c} + \tfrac{1}{4}\underset{\sim}{b}\cdot\underset{\sim}{b}\right) \\
&= 2\left(\tfrac{1}{2}\underset{\sim}{b}\cdot\underset{\sim}{b} + \tfrac{1}{2}\underset{\sim}{c}\cdot\underset{\sim}{c} - \underset{\sim}{a}\cdot\underset{\sim}{b} - \underset{\sim}{a}\cdot\underset{\sim}{c} + \underset{\sim}{a}\cdot\underset{\sim}{a}\right) \\
&= \underset{\sim}{b}\cdot\underset{\sim}{b} + \underset{\sim}{c}\cdot\underset{\sim}{c} - 2\underset{\sim}{a}\cdot\underset{\sim}{b} - 2\underset{\sim}{a}\cdot\underset{\sim}{c} + 2\underset{\sim}{a}\cdot\underset{\sim}{a} \\
&= \underset{\sim}{b}\cdot\underset{\sim}{b} - 2\underset{\sim}{a}\cdot\underset{\sim}{b} + \underset{\sim}{a}\cdot\underset{\sim}{a} + \underset{\sim}{c}\cdot\underset{\sim}{c} - 2\underset{\sim}{a}\cdot\underset{\sim}{c} + \underset{\sim}{a}\cdot\underset{\sim}{a} \\
&= (\underset{\sim}{b}-\underset{\sim}{a})\cdot(\underset{\sim}{b}-\underset{\sim}{a}) + (\underset{\sim}{c}-\underset{\sim}{a})\cdot(\underset{\sim}{c}-\underset{\sim}{a}) \\
&= \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{AC}\right|^2
\end{aligned}$$

Marking criteria	Marks
Correct solution	4
Makes significant progress towards proving the relationship	3
Determines $D, \overrightarrow{AD}$ and attempts to use $\lvert\underset{\sim}{v}\rvert^2 = \underset{\sim}{v}\cdot\underset{\sim}{v}$	2
Correctly determines at least one of $\overrightarrow{AB}, \overrightarrow{AC}$.	1

(c) i) Since L_1 and L_2 are perpendicular then $\begin{pmatrix}-2\\1\\-4\end{pmatrix}\cdot\begin{pmatrix}q\\2\\2\end{pmatrix}=0$

Hence $-2q+2-8=0 \;\Rightarrow\; q=-3$

Marking criteria	Marks
Correct proof	1

(c) ii) As L_1 and L_2 intersect, then the system of equations $\begin{cases}11-2\lambda=-5-3\mu & (1)\\ 2+\lambda=11+2\mu & (2)\\ 17-4\lambda=p+2\mu & (3)\end{cases}$ is consistent.

$(1)+2\times(2) \;\Rightarrow\; 15=17+\mu \;\Rightarrow\; \mu=-2$

From (2) $2+\lambda=11-4 \;\Rightarrow\; \lambda=5$ and from (3) $17-20=p-4 \;\Rightarrow\; p=1$

Marking criteria	Marks
Correct answer	3
Finds the value of λ or μ or equivalent merit.	2
Write a system of equations or equivalent merit.	1

(c) (iii) At the point of intersection $\begin{cases}x=11-2\lambda\\ y=2+\lambda\\ z=17-4\lambda\end{cases}$

Hence the lines intersect at $(1,7,-3)$.

Marking criteria	Marks
Correct answer	2
Find one of the coordinates of the point of intersection or equivalent merit.	1

(d) $$\int\frac{3dx}{x^2-6x+13}=\int\frac{3dx}{x^2-6x+9+4}$$

$$=3\int\frac{dx}{(x-3)^2+4}$$

$$=\frac{3}{2}\tan^{-1}\left(\frac{x-3}{2}\right)+C$$

Marking criteria	Marks
Correct answer	2
Obtaining $\int\frac{3dx}{x^2-6x+13}=3\int\frac{dx}{(x-3)^2+4}$ or equivalent merit.	1

Question 16

(a) $\left|2+2\sqrt{3}i\right| = \sqrt{(2)^2+\left(2\sqrt{3}\right)^2} = 4$

$\arg\left(2+2\sqrt{3}i\right) = \tan^{-1}\left(\frac{2\sqrt{3}}{2}\right) = \frac{\pi}{3}$

Let $\left(r(\cos\theta + i\sin\theta)\right)^4 = 4\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

Then $r^4 = 4 \;\Rightarrow\; r = \sqrt{2}$

$4\theta = 2k\pi + \frac{\pi}{3}$ for $k = -2, -1, 0, 1 \;\Rightarrow\; \theta = \frac{(6k+1)\pi}{12} = -\frac{11\pi}{12}, -\frac{5\pi}{12}, \frac{\pi}{12}, \frac{7\pi}{12}$

The roots are $\sqrt{2}\operatorname{cis}\left(-\frac{11\pi}{12}\right), \sqrt{2}\operatorname{cis}\left(-\frac{5\pi}{12}\right), \sqrt{2}\operatorname{cis}\left(\frac{\pi}{12}\right), \sqrt{2}\operatorname{cis}\left(\frac{7\pi}{12}\right)$

Marking criteria	Marks
Correct solution	4
Finds r or θ or equivalent merit.	3
Finds the modulus and argument of $2+2\sqrt{3}$ or equivalent merit.	2
Finds the modulus or argument of $2+2\sqrt{3}$ or equivalent merit.	1

(b) (i) $\overrightarrow{AC} = 8\underset{\sim}{i} - 4\underset{\sim}{j} - 4\underset{\sim}{k}$ and $\overrightarrow{BD} = -4\underset{\sim}{i} + (2-c)\underset{\sim}{j} - 12\underset{\sim}{k}$

If $\overrightarrow{AC} \perp \overrightarrow{BD}$ then $\left(8\underset{\sim}{i} - 4\underset{\sim}{j} - 4\underset{\sim}{k}\right)\cdot\left(-4\underset{\sim}{i} + (2-c)\underset{\sim}{j} - 12\underset{\sim}{k}\right) = 0$

i.e., $-32 - 4(2-c) + 48 = 0 \;\Rightarrow\; c = -2$

Marking criteria	Marks
Correct solution	2
Calculates $\overrightarrow{AC}$ or $\overrightarrow{BD}$ or equivalent merit	1

(b) ii) The line through A and C is $r_{AC} = -4\underset{\sim}{i} + 3\underset{\sim}{j} + 3\underset{\sim}{k} + \lambda\left(8\underset{\sim}{i} - 4\underset{\sim}{j} - 4\underset{\sim}{k}\right)$

$= (8\lambda - 4)\underset{\sim}{i} + (3-4\lambda)\underset{\sim}{j} + (3-4\lambda)\underset{\sim}{k}$

The line through B and D is $r_{BD} = -\underset{\sim}{i} - 2\underset{\sim}{j} + 6\underset{\sim}{k} + \mu\left(-4\underset{\sim}{i} + 4\underset{\sim}{j} - 12\underset{\sim}{k}\right)$

$= (4-4\mu)\underset{\sim}{i} + (4\mu - 2)\underset{\sim}{j} + (6-12\mu)\underset{\sim}{k}$

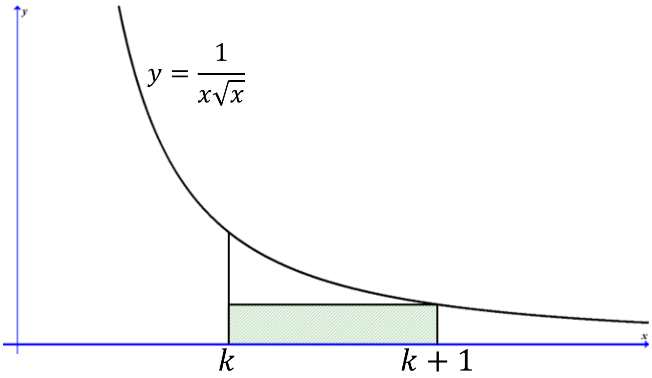
Since the $\underset{\sim}{j}$ and $\underset{\sim}{k}$ components of r_{AC} are always equal, the point of intersection requires $(4\mu - 2) = (6 - 12\mu) \;\Rightarrow\; \mu = \frac{1}{2}$.

This gives the point $2\underset{\sim}{i} + 0\underset{\sim}{j} + 0\underset{\sim}{k}$ on r_{BD}

This point also lies on r_{AC} with $\lambda = \frac{3}{4}$.

Marking criteria	Marks
Correct solution	3
Finds both vector equations and makes some progress towards E or equivalent merit	2
Correctly finds at least one of the two lines in parametric form or equivalent merit	1

(c) i)



Area of shaded rectangle is $\frac{1}{(k+1)\sqrt{k+1}}$

Hence $\frac{1}{(k+1)\sqrt{k+1}} < \int_k^{k+1} \frac{1}{x\sqrt{x}}\,dx$

$$= -\left[\frac{2}{\sqrt{x}}\right]_k^{k+1}$$

$$= \frac{2}{\sqrt{k}} - \frac{2}{\sqrt{k+1}}$$

Marking criteria	Marks
Correctly obtains inequality	2
Correct inequality or equivalent merit.	1

(c) ii) Let S_n be the statement that $\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \cdots + \frac{1}{n\sqrt{n}} < 3 - \frac{2}{\sqrt{n}}$.

At $n = 2$, LHS $= \frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}}$

$= 1.353553391 \ldots$

RHS $= 3 - \frac{2}{\sqrt{2}}$

$= 1.585786438 \ldots$

Hence S_2 is true.

Suppose that there is a k such that S_k is true. i.e., $\frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \cdots + \frac{1}{k\sqrt{k}} < 3 - \frac{2}{\sqrt{k}}$.

Then for $n = k+1$, LHS $= \frac{1}{1\sqrt{1}} + \frac{1}{2\sqrt{2}} + \frac{1}{3\sqrt{3}} + \cdots + \frac{1}{k\sqrt{k}} + \frac{1}{(k+1)\sqrt{k+1}}$

$< 3 - \frac{2}{\sqrt{k}} + \frac{1}{(k+1)\sqrt{k+1}}$

$< 3 - \frac{2}{\sqrt{k}} + \left(\frac{2}{\sqrt{k}} - \frac{2}{\sqrt{k+1}}\right)$ from part (i)

$= 3 - \frac{2}{\sqrt{k+1}}$

$=$ RHS

Hence if S_k is true then S_{k+1} is true and since S_2 is true then by the principle of mathematical induction, S_n is true $\forall n \geq 3, n \in \mathbb{Z}$.

Marking criteria	Marks
Correct proof.	4
Makes use of part (i) or equivalent merit.	3
Makes some progress towards an inductive style argument or equivalent merit.	2
Establishes the truth of S_2 or equivalent merit.	1